A Poincaré lemma in Geometric Quantisation

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Abstract

This paper presents a Poincaré lemma for the Kostant complex, used to compute geometric quantisation, when the polarisation is given by a Lagrangian foliation defined by an integrable system with nondegenerate singularities.

1 Introduction

The aim of this paper is to prove a Poincaré lemma for a complex that is used to compute geometric quantisation associated to a given real polarisation. It can be considered as the sequel of [15], in which the existence of a Poincaré lemma was investigated for a complex that computes the *foliated cohomology* of a foliated manifold.

In [15] we concluded that, if a foliation admits (special types of) singularities, then the foliated Poincaré lemma, which is well-known to hold for regular foliations, does no longer exist in general. The motivating example in [15] was the foliation determined by the Hamiltonian vector fields of an integrable system with nondegenerate singularities.

The reason to consider these set of examples in [15] comes from the fact that integrable systems provide natural examples of real polarisations. Real polarisations show up naturally in Geometric Quantisation, where additional data needs to be considered in order to choose a quantum representation space: the ingredients being a complex line bundle, a Hermitian connection

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and a polarisation (see [27] for more details on the theory of Geometric Quantisation).

The case of Geometric Quantisation with Kähler polarisation is a classical one. For some classical references see [8, 19, 27] and references therein.

It was probably Kostant [21, 19] who first suggested to define this representation space as the cohomology with coefficients in the sheaf of flat sections of the prequantum bundle (since in general no global sections exist in the case of real polarisations).

Explicit computations of geometric quantisation for real polarisations have been done in the case of fibration by tori [21] and for Gelfand-Cetlin systems [7]. In [7], finding an explicit set of action-angle coordinates plays an important rôle in the calculation (quantisation is given by the set where action coordinates take integral values, these, in turn, are the Bohr-Sommerfeld leaves of the system) and connects with the field of representation theory, since it computes the dimension of the spaces on which a representation of a prescribed maximal weight is given. It is important to point out here that singularities of Gelfand-Cetlin systems are excluded in this computation.

The recent paper [18] considers the general case of polarisations which are not necessarily given by a fibration and develops tools for dealing with sheaf cohomology computations in those cases, discussing also some *pathological* cases.

For real polarisation with singularities, these computations have been extended in the toric case [9, 22]. Also hyperbolic singularities are considered in [10].

One approach to compute this cohomology is à la de Rham, by finding a resolution of the sheaf of flat sections. A different approach using Čech cohomology can be found in [9] and [10].

Following Kostant [21, 19], a recipe for a resolution of the sheaf of sections can be obtained by *twisting* the sheaf computing foliated cohomology with the sheaf of flat sections. This observation allows to explicitely attack the problem of computation of Geometric Quantisation with real regular polarisations. In this case, this complex is a fine resolution of the sheaf of flat sections because of the existence of a Poincaré lemma for the foliated cohomology complex.

When this real polarisation has singularities this recipe does not hold in general, because it is no longer true that the foliated cohomology complex admits a Poincaré lemma; even if a singular Poincaré lemma had been proved for the deformation complex of integrable systems with nondegenerate singularities [16].

The purpose of this paper is to prove that, albeit the nonexistence of a Poincaré lemma for foliated cohomology in the singular case, we may still prove a Poincaré lemma for the twisted complex (called Kostant complex in the sequel) when the foliation is given by an integrable system with non-degenerate singularities. In particular, this allows to compute geometric quantisation by calculating the cohomology of the Kostant complex. In order to do so, we exploit the geometrical properties of the kind of singularities showing up. On the one hand, if the singularities are of elliptic or focus-focus type, the existence of circle actions with good properties allows to prove a Poincaré lemma [22].

If the singularities are of hyperbolic type we can still prove a Poincaré lemma using sharp analysis of Taylor flat functions. In this paper we give a complete proof of a Poincaré lemma for Kostant complex when there are hyperbolic singularities.

1.1 Organisation of this paper

In section 2 we recall the basics of prequantisation and some basic facts of integrable systems with singularities. In section 3 we talk about quantisation with real polarisations and the definition due to Kostant of Geometric Quantisation. In section 4 we describe the Lie pseudoalgebras approach to foliated cohomology necessary to deal with the Kostant complex in the singular case. In section 5 we provide a brief summary of the results contained in [15] and [22] concerning the foliated cohomology and the Kostant complex when circle actions are taken into account. In Section 6 we prove a Poincaré lemma for Geometric Quantisation with hyperbolic singularities. Finally, in Section 7 we consider the case of higher dimensions.

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2 Prequantisation

This section deals with some concepts needed to define wave functions. The first attempt to define a quantisation space was to see them as sections of a complex line bundle over the symplectic manifold, the so-called prequantum line bundle. The other notion described here, *polarisations*, is a way to define a global distinction between momentum and position.

2.1 Prequantum line bundle

A symplectic manifold (M, ω) such that the de Rham class $[\omega]$ is integral is called prequantizable. A prequantum line bundle of (M, ω) is a Hermitian line bundle over M with connection, compatible with the Hermitian structure, (L, ∇^ω) that satisfies $curv(\nabla^\omega) = -i\omega$ (the curvature of ∇^ω is proportional to the symplectic form).

Any exact symplectic manifold satisfies $[\omega] = 0$, in particular cotangent bundles with the canonical symplectic structure. In that case, the trivial line bundle is an example of a prequantum line bundle.

The following theorem[1] [12] provides a relation between the above definitions:

Theorem 2.1 (Kostant)**.** *A symplectic manifold (M, ω) admits a prequantum line bundle (L, ∇^ω) if and only if it is prequantisable.*

2.2 Polarisations given by nondegenerate integrable systems

An integrable system on a symplectic manifold (M, ω) of dimension $2n$ is a set of n functions $f_1, \ldots, f_n \in C^\infty(M)$ satisfying $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \neq 0$ over an open dense subset of M and $\{f_i, f_j\} = 0$ for all i, j. The mapping $F = (f_1, \ldots, f_n) : M \to \mathbb{R}^n$ is called a moment map.

The Poisson bracket is defined by $\{f, g\} = X_f(g)$, where X_f is the unique vector field defined by the equation $\imath_{X_f}\omega = -\mathrm{d}f$, called the Hamiltonian vector field of f.

The distribution generated by the Hamiltonian vector fields of the moment map, $\langle X_{f_1}, \ldots, X_{f_n} \rangle$, is involutive because $[X_f, X_g] = X_{\{f,g\}}$. Since $0 = \{f_i, f_j\} = \omega(X_{f_i}, X_{f_j})$, the leaves of the associated (possibly singular) foliation are isotropic submanifolds and they are Lagrangian at points where the functions are functionally independent.

A real polarisation $\mathcal{P}$ is an integrable distribution of TM in the Sussmann's sense [23] whose leaves are generically Lagrangian. The complexification of $\mathcal{P}$ is denoted by P and will be called polarisation. From now on (L, ∇^ω) will be a prequantum line bundle and P the complexification of it a real polarisation of (M, ω).

[1] This result is also attributed to André Weil, Introduction à l'étude des variétés kählériennes (1958).

There is a notion of nondegenerate singular points which was initially introduced by Eliasson [4, 5].

We denote by $(x_1, y_1, \ldots, x_n, y_n)$ a set of coordinates centered at the origin of $\mathbb{R}^{2n}$, and by ω the Darboux symplectic form $\omega = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$ in this neighborhood.

When p is a singular point for the moment map, since the functions f_i are in involution with respect to the Poisson bracket, the quadratic parts of the functions f_i commute, defining in this way an Abelian subalgebra of $Q(2n, \mathbb{R})$ (the set of quadratic forms on $2n$-variables). We say that these singularities are of nondegenerate type if this subalgebra is a Cartan subalgebra.

Cartan subalgebras of $Q(2n, \mathbb{R})$ were classified by Williamson in [26].

Theorem 2.2 (Williamson)**.** *For any Cartan subalgebra $\mathcal{C}$ of $Q(2n, \mathbb{R})$ there is a symplectic system of coordinates $(x_1, y_1, \ldots, x_n, y_n)$ in $\mathbb{R}^{2n}$ and a basis $h_1, \ldots, h_n$ of $\mathcal{C}$ such that each h_i is one of the following:*

$$\begin{array}{lcr}
h_i = x_i^2 + y_i^2 & \text{for } 1 \leq i \leq k_e \ , & \text{(elliptic)} \\
h_i = x_i y_i & \text{for } k_e + 1 \leq i \leq k_e + k_h \ , & \text{(hyperbolic)} \\
\begin{cases} h_i = x_i y_i + x_{i+1} y_{i+1}, \\ h_{i+1} = x_i y_{i+1} - x_{i+1} y_i \end{cases} & \begin{array}{c} \text{for } i = k_e + k_h + 2j - 1, \\ 1 \leq j \leq k_f \end{array} & \text{(focus-focus pair)}
\end{array} \tag{1}$$

Thus the number of elliptic components k_e, hyperbolic components k_h and focus-focus components k_h is an invariant of the algebra $\mathcal{C}$. The triple (k_e, k_h, k_f) with $n = k_e + k_h + 2k_f$ is an invariant of the singularity and it is called the Williamson type of $\mathcal{C}$. Let $h_1, \ldots, h_n$ be a Williamson basis of this Cartan subalgebra. We denote by X_i the Hamiltonian vector field of h_i with respect to ω. Those vector fields form a basis of the corresponding Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$. We say that a vector field X_i is hyperbolic (resp. elliptic) if the corresponding function h_i is so. We say that a pair of vector fields X_i, X_{i+1} define a focus-focus pair if X_i and X_{i+1} are the Hamiltonian vector fields associated to functions h_i and h_{i+1} in a focus-focus pair.

In the local coordinates specified above, the vector fields X_i take the following form:

- X_i is an elliptic vector field,

$$X_i = 2\left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right) \ ; \tag{2}$$

- X_i is a hyperbolic vector field,

$$X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \ ; \tag{3}$$

- X_i, X_{i+1} is a focus-focus pair,

$$X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_{i+1}} \tag{4}$$

and

$$X_{i+1} = -x_i \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_i} + x_{i+1} \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_{i+1}} \ . \tag{5}$$

Assume that $\mathcal{F}$ is a linear foliation on $\mathbb{R}^{2n}$ with a rank 0 singularity at the origin p. Assume that the Williamson type of the singularity is (k_e, k_h, k_f). The linear model for the foliation is then generated by the vector fields above, it turns out that these type of singularities are symplectically linearizable and we can read of the local symplectic geometry of the foliation from the algebraic data associated to the singularity (Williamson type).

This is the content of the following symplectic linearisation result in [4],[5] and [14],

Theorem 2.3. *Let ω be a symplectic form defined in a neighborhood U of the origin p for which $\mathcal{F}$ is generically Lagrangian, then there exists a local diffeomorphism $\phi : (U, p) \longrightarrow (\phi(U), p)$ such that ϕ preserves the foliation and $\phi^*(\sum_i \mathrm{d}x_i \wedge \mathrm{d}y_i) = \omega$, with x_i, y_i local coordinates on $(\phi(U), p)$.*

Furthermore, if $\mathcal{F}'$ is a generically Lagrangian foliation and has $\mathcal{F}$ as a linear foliation model near a point, one can symplectically linearise $\mathcal{F}'$ (see [14]).

This is equivalent to Eliasson's theorem [4, 5] in the completely elliptic case.

There are normal forms for higher rank which have been obtained by the first author together with Nguyen Tien Zung [14, 17] also in the case of singular nondegenerate compact orbits. When the rank of the singularity is greater than 0, a collection of regular vector fields is also attached to it. We can then reduce the k-rank case to the 0-rank case via a Marsden-Weinstein reduction associated to a natural Hamiltonian $\mathbb{T}^k$-action given by the joint flow of the moment map F.

3 Geometric quantisation à la Kostant

The original idea of geometric quantisation is to associate a Hilbert space to a symplectic manifold via a prequantum line bundle and a polarisation. Usually this is done using flat global sections of the line bundle, however, their existence is a nontrivial matter[2]. In case these global sections do not exist, Kostant suggested to consider higher cohomology groups, by taking cohomology with coefficients in the sheaf of flat sections, to define geometric quantisation.

Example 3.1. We consider the cotangent bundle of the circle endowed with the canonical symplectic structure as an explicit example of the nonexistence of nonzero global flat sections: $M = \mathbb{R} \times S^1$ with coordinates (x, y) and $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Take as L the trivial complex line bundle with connection 1-form $\Theta = x\mathrm{d}y$ and $\mathcal{P} = \langle \frac{\partial}{\partial y} \rangle$. Flat sections satisfy $\nabla^{\omega}_{\frac{\partial}{\partial y}} s = \mathrm{d}s(\frac{\partial}{\partial y}) - i\Theta(\frac{\partial}{\partial y})s = 0$. Thus $s(x, y) = f(x)e^{ixy}$, for some function f, and it has period 2π in y if and only if $x \in \mathbb{Z}$, for S^1 the unity circle. Thus the flat section is only well-defined for the set of points with $x \in \mathbb{Z}$.

Let $\mathcal{J}$ denotes the space of local sections, s, of a prequantum line bundle, L, such that $\nabla^{\omega}_X s = 0$ for all vector fields $X \in P$. The space $\mathcal{J}$ has the structure of a sheaf and it is called the sheaf of flat sections.

The quantisation of $(M, \omega, L, \nabla^{\omega}, P)$ is given by

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} \check{H}^k(M; \mathcal{J}) , \tag{6}$$

where $\check{H}^k(M; \mathcal{J})$ are Čech cohomology groups with values in the sheaf $\mathcal{J}$. In this case, one implicitly assumes to have prescribed all the extra structures and calls M a quantisable manifold.

Remark 3.1. Even though $\mathcal{Q}(M)$ is just a vector space and a priori has no Hilbert structure, it will be called quantisation. The true quantisation of the triplet $(M, \omega, L, \nabla^{\omega}, P)$ shall be the completion of the vector space $\mathcal{Q}(M)$, after a Hilbert structure is given, together with a Lie algebra homomorphism (possibly defined over a smaller set) between the Poisson algebra of $C^{\infty}(M)$ and operators on the Hilbert space. In spite of the problems that may exist in order to define geometric quantisation using $\mathcal{Q}(M)$, the first step is to compute this vector space.

[2] Actually Rawnsley [19] (also [22], under slightly different hypotheses) showed that the existence of a S^1-action may be an obstruction for the existence of nonzero global flat sections

4 Lie pseudoalgebras and the Kostant complex

Instead of computing directly the Čech cohomology groups $\check{H}^k(M;\mathcal{J})$, the strategy is to present a resolution for the sheaf $\mathcal{J}$. For regular polarisations this has been done by Kostant [21, 19]. In the singular case this can be achieved via Lie pseudoalgebra representations. This section only recasts geometric quantisation under the language of Lie pseudoalgebras and its representations, the proof that the Kostant complex is a resolution for the sheaf is left to the last two sections.

The set $C^\infty(M)$ is a commutative $\mathbb{C}$-algebra and the polarisation induced by a integrable system $F: M \to \mathbb{R}^n$ on (M,ω), $(P = \langle X_1,\dots,X_n\rangle_{C^\infty(M)}, [\cdot,\cdot]\big|_P)$, where X_i is the Hamiltonian vector field of the ith component the moment map, is both a $C^\infty(M)$-module and a $\mathbb{C}$-Lie algebra. The Lie algebra and $C^\infty(M)$-module structures are compatible in such a way that $(P, C^\infty(M), \mathbb{C})$ is an example of a Lie pseudoalgebra (see [13] for precise definitions and a nice account for the history and, various, names of this structure).

Considering $C^\infty(M)$ as a $C^\infty(M)$-module, $(P,[\cdot,\cdot]\big|_P)$ can be represented on $C^\infty(M)$ as vector fields acting on smooth functions. Thus, one can consider the following complex (Lie pseudoalgebra cohomology):

$$0 \longrightarrow C^\infty_P(M) \hookrightarrow C^\infty(M) \xrightarrow{\mathrm{d}_P} \Omega^1_P(M) \xrightarrow{\mathrm{d}_P} \cdots \xrightarrow{\mathrm{d}_P} \Omega^n_P(M) \xrightarrow{\mathrm{d}_P} 0\ , \tag{7}$$

With the differential defined by

$$\begin{aligned}\mathrm{d}_P\alpha(Y_1,\dots,Y_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}Y_i(\alpha(Y_1,\dots,\hat{Y}_i,\dots,Y_{k+1})) \\ &\quad + \sum_{i<j}(-1)^{i+j}\alpha([Y_i,Y_j],Y_1,\dots,\hat{Y}_i,\dots,\hat{Y}_j,\dots,Y_{k+1})\end{aligned} \tag{8}$$

and $C^\infty_P(M) = \ker(\mathrm{d}_P)$, $\Omega^k_P(M) = \mathrm{Hom}_{C^\infty(M)}(\bigwedge^k_{C^\infty(M)} P; C^\infty(M))$, $Y_1,\dots,Y_{k+1} \in P$.

The differential is a coboundary operator and the associated cohomology is denoted by $H_P^\bullet(M)$.

Remark 4.1. Only when the polarisation is regular one has a honest Lie algebroid structure, otherwise one really has a Lie pseudoalgebra.

Proposition 4.1. *The restriction of the connection ∇^ω to the polarisation, $\nabla := \nabla^\omega\big|_P$, defines a representation of the Lie pseudoalgebra $(P, C^\infty(M), \mathbb{C})$ on $\Gamma(L)$.*

Proof. The space of sections of the prequantum line bundle L is clearly a $C^\infty(M)$-module, and

$$\nabla : \Gamma(L) \to \Omega^1_P(M) \otimes_{C^\infty(M)} \Gamma(L) \tag{9}$$

satisfies (by definition) the following property:

$$\nabla(fs) = \mathrm{d}_P f \otimes s + f\nabla s \ , \tag{10}$$

for any $f \in C^\infty(M)$ and $s \in \Gamma(L)$.

If $X, Y \in P$, thinking of ∇ as a linear map from P to endomorphisms of $\Gamma(L)$,

$$\nabla_{[X,Y]} = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - curv(\nabla)(X,Y) \ . \tag{11}$$

But since $curv(\nabla^\omega) = -i\omega$ vanishes along P, $curv(\nabla)(X,Y) = 0$ and ∇ is a Lie algebra representation of $(P, [\cdot,\cdot]\big|_P)$ on $\Gamma(L)$ compatible with their $C^\infty(M)$-module structures. □

With respect to line bundle valued polarised forms, i.e., elements of $S_P{}^\bullet(L) = \bigoplus_{k\geq 0} S^k_P(L)$, where $S^k_P(L) = \Omega^k_P(M) \otimes_{C^\infty(M)} \Gamma(L)$, the previous proposition asserts that the degree $+1$ map $\mathrm{d}^\nabla : S_P{}^\bullet(L) \to S_P{}^\bullet(L)$, defined by

$$\mathrm{d}^\nabla(\alpha \otimes s) = \mathrm{d}_P\alpha \otimes s + (-1)^{\mathrm{degree}(\alpha)} \alpha \wedge \nabla s \ , \tag{12}$$

is a coboundary.

Thus, the associated Lie pseudoalgebra cohomology of this representantion, $H^\bullet(S_P{}^\bullet(L))$, induces a complex (on the sheaf level). This complex is called the Kostant complex and is defined as,

$$0 \longrightarrow \mathcal{J} \hookrightarrow \mathcal{S} \xrightarrow{\nabla} \mathcal{S}^1_P(L) \xrightarrow{\mathrm{d}_\nabla} \cdots \xrightarrow{\mathrm{d}_\nabla} \mathcal{S}^n_P(L) \xrightarrow{\mathrm{d}_\nabla} 0 \ , \tag{13}$$

with $\mathcal{S}$ denoting the sheaf of sections of the line bundle L and $\mathcal{S}^k_P(L)$ the sheaves associated to $\Omega^k_P(M) \otimes_{C^\infty(M)} \Gamma(L)$.

Remark 4.2. The only property of L being used in this paper is the existence of flat connections along P; any complex line bundle would do, not only a prequantum one —the results here work if metaplectic correction is included (see [27] for details about the rôle of metaplectic correction in Geometric Quantisation).

5 The de Rham foliated complex versus the Kostant complex

In this section, a proof that the Kostant complex is a fine resolution for the sheaf of flat sections is presented for regular polarisations. Other results involving nondegenerate singularities and the existence or nonexistence of a Poincaré lemma for the de Rham foliated complex and the Kostant complex are also discussed.

The following result uses the foliated Poincaré lemma for regular foliations. A small account of the foliated Poincaré lemma can be found in [15]. The following result is also reproduced in [15] with a slightly different proof.

Lemma 5.1. *Given a regular polarisation P there always exists a local unitary flat section on each point of M.*

Proof. The symplectic form is closed, $\mathrm{d}\omega = 0$, thus locally $\omega = \mathrm{d}\theta$ and, since P is Lagrangian, ω vanishes in the directions tangent to the leaves of P; which implies $\mathrm{d}_P\Theta = 0$, where Θ is the restriction of θ in the directions tangent to the leaves of the polarisation. By the foliated Poincaré lemma, there exists a function f such that $\mathrm{d}_P f = \Theta$. Therefore $\theta - \mathrm{d}f$ satisfies $\mathrm{d}(\theta - \mathrm{d}f) = \omega$ and $(\theta - \mathrm{d}f)$ vanishes in the directions tangent to the leaves.

Since at any subset of M where ω is the differential of a 1-form there exists a unitary section such that its associated potential is this particular 1-form, one has a unitary section s satisfying $\nabla_X s = -i[\theta - \mathrm{d}f](X)s$, which is a flat section: $\nabla_X s = 0$ for any $X \in \Gamma(P)$ because $(\theta - \mathrm{d}f)$ vanishes in the directions tangent to the leaves. $\square$

As a consequence of the existence of unitary flat sections, elements of $\mathcal{S}_P^k(L)$ which are closed can be interpreted as the germ of closed polarised k-forms taking values in the sheaf $\mathcal{J}$: locally, in a trivialising neighbourhood of L with a unitary flat section s, a k-form $\alpha \otimes s$ is closed if and only if $\mathrm{d}_P\alpha = 0$, because $\mathrm{d}_\nabla(\alpha \otimes s) = \mathrm{d}_P\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$, $s \neq 0$ and $\nabla s = 0$. Wherefore, together with the foliated Poincaré lemma, lemma 5.1 implies the exactness of (13).

The sheaves $\mathcal{S}_P^k(L)$ are fine: $\Gamma(L)$ and $\Omega_P^k(M)$ are free modules over the ring of functions of M, and because of that, they admit partition of unity. Hence, via a Poincaré lemma, the abstract de Rham theorem [3] entails [21, 19]:

Theorem 5.1 (Kostant)**.** *The Kostant complex is a fine resolution of $\mathcal{J}$. Therefore each of its cohomology groups, $H^k(S_P{}^\bullet(L))$, is isomorphic to $\check{H}^k(M;\mathcal{J})$.*

It is important to notice that the proof of lemma 5.1 relies on the existence of a Poincaré lemma for foliations. When the foliation is not regular such theorem might not exist, and the proof of lemma 5.1 if of no use; therefore, one needs a different method to prove that the Kostant complex is a fine resolution for the sheaf of flat sections.

This is exactly the situation for polarisations induced by nondegenerate integrable systems, for which we proved in [15] that there is no Poincaré lemma for the foliated complex.

Theorem 5.2 (Miranda and Solha). *The foliated Poincaré lemma does not hold for foliations defined by integrable systems with nondegenerate singularities of Williamson type* $(k_e, k_h, 0)$.

In [15] we explicitly computed the cohomology groups in some instances —in particular degree 1 and top degree for smooth systems and in all the degrees for analytic ones. Thus, in order to prove a Poincaré lemma for the Kostant complex, different strategies need to be adopted. Luckily, it is possible to prove Poincaré lemmata for the Kostant complex when almost toric nondegenerate singularities are included in the picture.

The following result is contained in [22]:

Theorem 5.3 (Solha). *The Kostant complex is a fine resolution for* $\mathcal{J}$ *when* $\mathcal{P}$ *is given by a locally toric singular Lagrangian fibration or an almost toric fibration in dimension* 4.

The proof of this theorem (corollary 6.1, proposition 6.4 and results in subsection 6.8 of [22]) is based on the existence of symplectic circle actions. Hyperbolic singularities do not share the same kind of symmetry as elliptic or focus-focus, i.e.: there is no natural symplectic circle action near purely hyperbolic singularities. Thus, again, the proof cannot be adapted to include hyperbolic singularities, and the next section is devoted to prove a Poincaré lemma for this remaining case.

6 A Poincaré lemma for the Kostant complex with hyperbolic singularities

We start this section fixing some notation for hyperbolic singularities. Let $(M = \mathbb{C}, \omega = \mathrm{d}x \wedge \mathrm{d}y)$ and $h : M \to \mathbb{R}$ be a nondegenerate integrable system of hyperbolic type, i.e.: $h(x, y) = xy$. For this case, the real polarisation is $\mathcal{P} = \langle X \rangle$, with X the Hamiltonian vector field $-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

(M, ω) is an exact symplectic manifold and the trivial line bundle is a prequantum line bundle for it: $L = \mathbb{C} \times \mathbb{C}$ with connection 1-form $\Theta = \frac{1}{2}(x\mathrm{d}y - y\mathrm{d}x)$ with respect to the unitary section e^{ih}.

Consider a section $f\mathrm{e}^{ih}$ of the prequantum line bundle, the flat section equation can be written as,

$$\nabla f\mathrm{e}^{ih} = 0 \;\Leftrightarrow\; X(f) = ihf \;. \tag{14}$$

This equation has been studied in [10]. Let us recall here proposition 3.5 of that paper,

Proposition 6.1 (Hamilton and Miranda)**.** *Any leafwise flat section σ defined over Z can be written as a collection*

$$\sigma_j = a_j(xy)\mathrm{e}^{\frac{i}{2}xy\ln|\frac{x}{y}|} \qquad j = 1, 2, 3, 4 \tag{15}$$

where a_j is a complex-valued smooth function of one variable, analytically flat at 0, with domain such that $a_j(xy)$ is defined on the j^{th} open quadrant of $\mathbb{R}^2$. Conversely, given four such a_j, they fit together to define a leafwise flat section σ over Z using the formula above.

Thus (up to a different choice of sign) this implies that

$$f(x,y)\mathrm{e}^{ixy} = \begin{cases} 0 & \text{if } x = 0, y = 0 \\ a_1(xy)\mathrm{e}^{\frac{i}{2}xy\ln\frac{y}{x}} & \text{if } x > 0, y > 0 \\ a_2(xy)\mathrm{e}^{\frac{i}{2}xy\ln\frac{-y}{x}} & \text{if } x > 0, y < 0 \\ a_3(xy)\mathrm{e}^{\frac{i}{2}xy\ln\frac{y}{-x}} & \text{if } x < 0, y > 0 \\ a_4(xy)\mathrm{e}^{\frac{i}{2}xy\ln\frac{y}{x}} & \text{if } x < 0, y < 0 \;, \end{cases} \tag{16}$$

where a_j is a smooth complex-valued function of one variable (defined for $z \in [0, \infty)$ if $j = 1, 4$ or $z \in (-\infty, 0]$ if $j = 2, 3$) and such that $\frac{\mathrm{d}^k a_j}{\mathrm{d}z^k}(0) = 0$ for all j and k.

The converse of proposition 6.1 guarantees that $H^0(S_P{}^{\bullet}(L))$ is not trivial and is given by quadruples of Taylor flat smooth complex valued functions of one variable, as above.

We also need the following property of polarised forms (proposition 5.1 in [15]),

Proposition 6.2. *If α is a polarised k-form, $\alpha \in \Omega^k_P(\mathbb{R}^{2n})$, then*

$$\alpha(X_{j_1}, \ldots, X_{j_k})\big|_{\Sigma_{j_1}\cup\cdots\cup\Sigma_{j_k}} = 0 \;. \tag{17}$$

With $\Sigma_i = \{p \in \mathbb{R}^{2n} \;;\; x_i(p) = y_i(p) = 0\}$ denoting the vanishing set of a vector field of a Williamson basis X_i.

For the computation of the first cohomology group the strategy is going to be close to the one used in [14, 16]: firstly, a formal solution is obtained and then a closed formula is given for the case of flat functions.

A 1-form $\alpha \otimes \mathrm{e}^{ih} \in S^1_P(L)$ is exact if and only if there exists a $g \in C^\infty(\mathbb{C})$ satisfying

$$\nabla g\mathrm{e}^{ih} = \alpha \otimes \mathrm{e}^{ih} \;\Leftrightarrow\; X(g) = ihg + \alpha(X) \; . \tag{18}$$

We denote the Taylor series in (x, y) of g and $\alpha(X)$ near the origin $(0, 0) \in \mathbb{C}$ by,

$$\sum_{k,\,l\,=\,0}^{\infty} g_{k,l} x^k y^l \tag{19}$$

and the Taylor series of $\alpha(X)$ by

$$\sum_{k,\,l\,=\,0}^{\infty} f_{k,l} x^k y^l \tag{20}$$

with $f_{0,0} = 0$, by definition (proposition 6.2).

The cohomological equation in jets, then, reads

$$\sum_{k,\,l\,=\,0}^{\infty} (l-k) g_{k,l} x^k y^l = \sqrt{-1} \sum_{k,\,l\,=\,0}^{\infty} g_{k,l} x^{k+1} y^{l+1} + \sum_{k,\,l\,=\,0}^{\infty} f_{k,l} x^k y^l \; . \tag{21}$$

And the following recursive relations lead to a solution,

$$\begin{aligned}
g_{0,0} &= 0 \; ; \\
g_{k,k} &= \sqrt{-1} f_{k+1,k+1} \; , && k > 0 \; ; \\
g_{0,k} &= \frac{f_{0,k} + \sqrt{-1} g_{0,k-1}}{k} \; , && k > 0 \; ; \\
g_{k,0} &= \frac{-f_{k,0} - \sqrt{-1} g_{k-1,0}}{k} \; , && k > 0 \; ; \\
g_{k,l} &= \frac{f_{k,l} + \sqrt{-1} g_{k-1,l-1}}{l-k} \; , && k \neq l > 0 \; .
\end{aligned} \tag{22}$$

We can even write a closed-form expression for the jets,

$$
\begin{aligned}
&g_{0,0} = 0 \ ; \\
&g_{k,k} = \sqrt{-1} f_{k+1,k+1} \ , && k > 0 \ ; \\
&g_{0,k} = \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^{\frac{j}{2}} (k-j-1)! f_{0,k-j} \ , && k > 0 \ ; \\
&g_{k,0} = \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^{\frac{j}{2}+1} (k-j-1)! f_{k-j,0} \ , && k > 0 \ ; \\
&g_{k,l} = \sum_{j=0}^{k-1} \frac{(-1)^{\frac{j}{2}}}{(l-k)^{j+1}} f_{k-j,l-j} + \sum_{j=0}^{l-k-1} \frac{(-1)^{\frac{k}{2}+\frac{j}{2}} (l-k-j-1)!}{(l-k)^k (l-k)!} f_{0,l-k-j} \ , && l > k > 0 \ ; \\
&g_{k,l} = \sum_{j=0}^{l-1} \frac{(-1)^{\frac{j}{2}}}{(l-k)^{j+1}} f_{k-j,l-j} + \sum_{j=0}^{k-l-1} \frac{(-1)^{\frac{l}{2}+\frac{j}{2}+1} (k-l-j-1)!}{(l-k)^l (k-l)!} f_{k-l-j,0} \ , && k > l > 0 \ .
\end{aligned}
\tag{23}
$$

This procedure solves the equation only formally. According to Borel's theorem [20], there exists, up to Taylor flat functions[3] at the origin, a unique smooth function with such Taylor series.

Thus, we have proved the following:

Lemma 6.1. *Any smooth function $\tilde{g}$ whose Taylor series is defined by the previous recursive relations satisfies,*

$$X(\tilde{g}) - ih\tilde{g} - \alpha(X) = F \ , \tag{24}$$

where F is a Taylor flat function at the origin.

Therefore, if it is possible to find a solution for

$$X(G) - ihG = F \ , \tag{25}$$

such that G is Taylor flat at the origin, the combination of functions $g = \tilde{g} - G$ defines a smooth solution for the cohomological equation.

One can solve this problem with the aid of the logarithmic function $\ln \gamma : \{(x, y) \in \mathbb{C} \ ; \ xy \neq 0\} \to \mathbb{R}$, where $\ln \gamma(p)$ is the time that it takes for a

[3]Observe that two smooth functions which have the same Taylor expand at a point differ by a smooth function which has vanishing jet at all order at that point.

point in the diagonal, $\{(x,y)\in\mathbb{C}\ ;\ x=y\}$, to reach p via the flow of X (the diagonal point and p lie over the same integral curve of X). This function is well defined for $xy\neq 0$.

Lemma 6.2. *For a given Taylor flat function F, a solution to the equation $X(G)-ihG=F$ is given by,*

$$G=\int_{-\ln\gamma}^{0}\mathrm{e}^{-iht}F\circ\phi_t\ \mathrm{d}t\ . \tag{26}$$

This solution is actually well defined and smooth over all points of $\mathbb{C}$.

Remark 6.1. Observe that the smoothness of this formula prevails if parameters are considered in the function F. This observation will be needed in the higher dimensional discussion.

Proof. Before proving that the expression for G is smooth and well defined, let us prove that G solves the equation by computing $X(G)$. We first consider the composition of G with the flow of X at time s,

$$G\circ\phi_s=\int_{-\ln\gamma\,\circ\,\phi_s}^{0}\mathrm{e}^{-ith\circ\phi_s}F\circ\phi_t\circ\phi_s\ \mathrm{d}t=\int_{-\ln\gamma\,-\,s}^{0}\mathrm{e}^{-ith}F\circ\phi_{t+s}\ \mathrm{d}t\ . \tag{27}$$

The logarithmic function satisfies $\ln\gamma\circ\phi_s=\ln\gamma+s$ and $h\circ\phi_s=h$, thus, by a change of coordinates $\tau=t+s$:

$$G\circ\phi_s=\int_{-\ln\gamma\,-\,s\,+\,s}^{s}\mathrm{e}^{-ih(\tau-s)}F\circ\phi_\tau\ \mathrm{d}\tau=\mathrm{e}^{ish}\int_{-\ln\gamma}^{s}\mathrm{e}^{-ith}F\circ\phi_t\ \mathrm{d}t\ . \tag{28}$$

Then, one differentiates $G\circ\phi_s$ with respect to s

$$\frac{\mathrm{d}}{\mathrm{d}s}G\circ\phi_s=ih\mathrm{e}^{ish}\int_{-\ln\gamma}^{s}\mathrm{e}^{-ith}F\circ\phi_t\ \mathrm{d}t+F\circ\phi_s\ , \tag{29}$$

and finally evaluate it in $s=0$ to get

$$X(G)=\left.\frac{\mathrm{d}}{\mathrm{d}s}G\circ\phi_s\right|_{s=0}=ih\int_{-\ln\gamma}^{0}\mathrm{e}^{-ith}F\circ\phi_t\ \mathrm{d}t+F=ihG+F\ . \tag{30}$$

It is clear that G is smooth and well defined over the points where the logarithmic function $\ln\gamma$ is well defined (the set $\{(x,y)\in\mathbb{C}\ ;\ xy\neq 0\}$). The idea now is to prove that it is continuous and well defined at the points where $h=0$.

For each point of $\{(x,y) \in \mathbb{C} \ ; \ xy \neq 0\}$,

$$|G| = \left| \int_{-\ln\gamma}^{0} \mathrm{e}^{-iht} F \circ \phi_t \ \mathrm{d}t \right| \leq \left| \int_{-\ln\gamma}^{0} \mathrm{e}^{-iht} \ \mathrm{d}t \right| \max_{t\in[-\ln\gamma,0]} |F \circ \phi_t| \ , \tag{31}$$

but

$$\left| \int_{-\ln\gamma}^{0} \mathrm{e}^{-iht} \ \mathrm{d}t \right| \max_{t\in[-\ln\gamma,0]} |F \circ \phi_t| = \left| \frac{\mathrm{e}^{ih\ln\gamma} - 1}{-ih} \right| \max_{t\in[-\ln\gamma,0]} |F \circ \phi_t| \ , \tag{32}$$

thus,

$$|G| \leq \left| \frac{\mathrm{e}^{ih\ln\gamma} - 1}{-ih} \right| \max_{t\in[-\ln\gamma,0]} |F \circ \phi_t| \ . \tag{33}$$

When h approaches zero, $\frac{\mathrm{e}^{ih\ln\gamma}-1}{-ih}$ diverges in a logarithmic fashion. It is left to understand how $\max\limits_{t\in[-\ln\gamma,0]} |F \circ \phi_t|$ behaves.

At a point $p = (x,y) \in \mathbb{C}$, the flow of the Hamiltonian vector field $X = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is given by $\phi_t(p) = (\mathrm{e}^{-t}x, \mathrm{e}^{t}y)$. Let $p_0 = (z,z)$ be a point of $\mathbb{C}$ satisfying $\phi_t(p_0) = p$, then

$$\ln\gamma(p) = \begin{cases} \frac{1}{2}\ln\dfrac{y}{x} & \text{if } xy > 0 \\ \frac{1}{2}\ln\dfrac{-y}{x} & \text{if } xy < 0 \end{cases} \ , \tag{34}$$

since

$$\mathrm{e}^{-t}z = x \ \Rightarrow \ t = \ln\frac{z}{x} \tag{35}$$

and

$$\mathrm{e}^{t}z = y \ \Rightarrow \ t = \ln\frac{y}{z} \ . \tag{36}$$

Wherefore,

$$\phi_{-\ln\gamma(p)}(p) = (|h(p)|^{\frac{1}{2}}, |h(p)|^{\frac{1}{2}}) \ , \tag{37}$$

which implies $\lim\limits_{|h|\to 0} F \circ \phi_{-\ln\gamma} = 0$ and it goes sufficiently fast to zero to guarantee that G is continuous and vanishes at $h = 0$, because the function F is Taylor flat at the origin.

One can see that G is actually smooth at $h = 0$ by analising its differential (it is clear that the argument that follows holds for the higher order partial derivatives):

$$\mathrm{d}G = \int_{-\ln\gamma}^{0} \left(\mathrm{e}^{-iht} {\phi_t}^{*} \circ \mathrm{d}F \right) \ \mathrm{d}t - iG\mathrm{d}h + \mathrm{e}^{-ih\ln\gamma} F \circ \phi_{-\ln\gamma} \mathrm{d}\ln\gamma \ . \tag{38}$$

The first term converges to zero, as h approaches to zero, by the same argument used above, the partial derivatives of a Taylor flat function are still Taylor flat by definition. The second term is continuous and well defined at $h = 0$ because G is and h is smooth. It remains to analise the term $F \circ \phi_{-\ln\gamma} \mathrm{d} \ln\gamma$. By l'Hôpital's rule $\lim_{h\to 0} \mathrm{e}^{-ih\ln\gamma} = 1$ and the fact that F is Taylor flat guarantees that $\lim_{h\to 0} F \circ \phi_{-\ln\gamma} \mathrm{d} \ln\gamma = 0$.

□

Since the dimension of the generic leaves is 1, the only cohomology group to check is the first cohomology group.

Lemmata 6.1 and 6.2 yield the following,

Theorem 6.1. *The first cohomology group $H^1(S_P^\bullet(L))$ vanishes when the polarisation that we consider is given by an integrable system on a two-dimensional manifold in a neighbourhood of a hyperbolic singularity.*

7 Higher dimensions

In this section we outline how to prove the Poincaré lemma for dimension greater than 2. The idea is to use the symplectic local model of integrable systems guaranteed by theorem 2.3, that enables to do computations which entail the reduction to the 2-dimensional case (or 4-dimensional when there are focus-focus singularities).

7.1 Künneth formulae

Consider a product of two prequantisable symplectic manifolds (M_1, ω_1) and (M_2, ω_2) endowed with nonsingular real polarisations $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. Observe that the product foliation $\mathcal{P}_1 \times \mathcal{P}_2$ is a Lagrangian foliation in the product symplectic manifold $(M_1 \times M_2, \omega_1 \oplus \omega_2)$.

We denote by $\mathcal{J}_1$ and $\mathcal{J}_2$ the sheaf of flat sections of the respective prequantisable line bundles.

One can relate the geometric quantisation of the product manifold with the geometric quantisation of the factors via a Künneth-type formula.

In [18] the first author of this paper together with Francisco Presas proved,

Theorem 7.1 (Miranda and Presas)**.** *There is an isomorphism,*

$$\check{H}^n(M_1 \times M_2; \mathcal{J}_{12}) \backsimeq \bigoplus_{p+q=n} \check{H}^p(M_1; \mathcal{J}_1) \otimes \check{H}^q(M_2; \mathcal{J}_2) \ , \tag{39}$$

whenever the Geometric Quantisation associated to $(M_1, \mathcal{J}_1)$ has finite dimension, M_1 is compact and M_2 admits a good *covering.*

When one allows the foliations to have singularities, proving such a formula equivalent becomes a tricky question since these topological spaces are sometimes of infinite dimension (like in the hyperbolic case [10]).

There is a more general Künneth theory for sheaves that can be handled in some cases to give a direct proof of some interesting facts of sheaf cohomology. Künneth formula for sheaves has been studied by many authors (see for instance [11]), and the notion of "completion" of the topological tensor product is used (this probably dates back to Grothendieck's thesis [6]).

In order to consider a completion of the cohomology group, a topology needs to be induced in the cohomology groups with coefficients in the sheaf of flat sections. In the regular case this topology is quite intuitive. For foliated cohomology we consider the de Rham complex and this yields an induced topological structure using the topology of the space of forms and the fact that for differential forms the exterior derivative d is continuous. In the case of general Fréchet sheaves a similar strategy can be used ([6] and [1]).

We recall here a result among a set of results in this direction proved in [11].

Theorem 7.2 (Kaup)**.** *For Fréchet sheaves $\mathcal{F}$ and $\mathcal{G}$ over compact M and N the following formula holds:*

$$\check{H}^n(M \times N; \mathcal{F} \times \mathcal{G}) = \bigoplus_{p+q=n} \check{H}^p(M; \mathcal{F}) \hat{\otimes} \check{H}^q(N; \mathcal{G}) \ , \tag{40}$$

whenever,

1. *the cohomology groups are Hausdorff;*
2. *either $\mathcal{F}$ or $\mathcal{G}$ are nuclear.*

This approach using nuclear spaces has also been adopted by Bertelson [1] in the case in which the considered sheaf is the one used in foliated cohomology.

One could try to apply Kaup's theorem to prove Poincaré lemma for the Kostant complex from the two-dimensional case [4](in case there are no focus-focus fibres) or from the combination of 2 and 4-dimensional cases in

[4]This approach may seem, a priori, quite naïve since for de Rham cohomology the proof of a Künneth formula for compact manifolds precisely uses as a first step for its proof the local case in which the Künneth formula prevails precisely because of Poincaré lemma (see [2]). In this section we are just using this formula to motivate the construction that will follow, even if this type formula combined with a Mayer-Vietoris-like argument is extremely useful to compute geometric quantisation of actual compact manifolds as it was seen in [18].

general, whenever such a result can be adapted for closed balls as it was done in Proposition 3.7 in [18]. We prefer to avoid this approach here and we rather provide a direct proof of this fact.

In any event, let us quickly sketch how a proof with an equivalent of Kaup's result with the appropriate hypothesis would work.

So let us assume that the conditions of the theorem are fulfilled (the Hausdorff and nuclear conditions do not seem to be a problem in the case of integrable systems with nondegenerate singularities) and let us adapt the notation of Kaup's theorem to our notation in this paper to obtain,

$$\check{H}^n(U_1 \times M_2; \mathcal{J}_{12}) = \bigoplus_{p+q=n} \check{H}^p(U_1; \mathcal{J}_1) \hat{\otimes} \check{H}^q(U_2; \mathcal{J}_2) \ , \tag{41}$$

The completion of the tensor product can be tricky sometimes but we just need to observe the following: Let us consider the case in which U_1 and U_2 are neighbourhoods. Since we have computed the factors $\check{H}^p(U_1; \mathcal{J}_1)$ and $\check{H}^q(U_2; \mathcal{J}_2)$ in the formula above and they all vanish if p and q are different from zero and this yields the vanishing of $\check{H}^n(U_1 \times U_2; \mathcal{J}_{12})$. This also entails the Poincaré lemma for general neighbourhoods since theorem 2.3 guarantees that we can assume that the neighbourhood is a product one symplectically and thus it is also the case of its connection.

This gives a *moral* proof of why this Poincaré lemma holds (even if one would need to verify that hypothesis are fulfilled and adapt Kaup's theorem). Let us give a down-to-earth proof by explicitely computing these cohomology groups with hands-on analysis.

This is what we do in the two sections below. We start by considering the hyperbolic-hyperbolic case and then we sketch how to prove the general case.

7.2 The hyperbolic-hyperbolic case

Let $(M_1 \times M_2 = \mathbb{C}^2, \omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$ and $H = (h_1, h_2) : M_1 \times M_2 \to \mathbb{R}$ be a nondegenerate integrable system of hyperbolic-hyperbolic type, i.e.: $H(x_1, y_1, x_2, y_2) = (x_1 y_1, x_2 y_2)$. For this case, the real polarisation is $\mathcal{P} = \langle X_1, X_2 \rangle$, with X_j the Hamiltonian vector field $-x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}$.

$(M_1 \times M_2, \omega)$ is an exact symplectic manifold and the trivial line bundle is a prequantum line bundle for it: $L = \mathbb{C} \times \mathbb{C}^2$ with connection 1-form $\Theta = \frac{1}{2}(x_1 \mathrm{d}y_1 - y_1 \mathrm{d}x_1) + \frac{1}{2}(x_2 \mathrm{d}y_2 - y_2 \mathrm{d}x_2)$ with respect to the unitary section $s = \mathrm{e}^{i(h_1 + h_2)}$.

To have a Poincaré lemma, one needs to prove that $H^1(S_P^\bullet(L))$ and $H^2(S_P^\bullet(L))$ are trivial[5]. Let us start with $H^2(S_P^\bullet(L))$, the easiest case.

[5]The cohomomlogy group $H^0(S_P^\bullet(L))$ can also be computed by a parametric version of

Proposition 7.1. *The second cohomology group $H^2(S_P^\bullet(L))$ vanishes for hyperbolic-hyperbolic singularities.*

Proof. Any line bundle valued polarised 2-form, $\alpha \otimes s$, is automatically closed in dimension 4, and it is exact if and only if there exists a β such that,

$$\mathrm{d}^\nabla(\beta \otimes s) = \alpha \otimes s \ . \tag{42}$$

Because $\nabla s = -i\Theta \otimes s$, the exactness of $\alpha \otimes s$ is equivalent to

$$\alpha = \mathrm{d}_P\beta - i\Theta \wedge \beta \ \Leftrightarrow \ \alpha(X_1, X_2) = X_1(\beta(X_2)) - ih_1\beta(X_2) - X_2(\beta(X_1)) + ih_2\beta(X_1) \ . \tag{43}$$

One can find a solution for this equation by taking $\beta(X_1) = 0$ and solving for $\beta(X_2)$, using the parametric versions of lemmata 6.1 and 6.2. This ends the proof of the proposition.

□

In order to compute $H^1(S_P^\bullet(L))$ one needs to prove parametric versions of lemmata 6.1 and 6.2 when the unknown functions posses a special property if the known functions have it. Concretely:

Lemma 7.1. *If $X_1(f) = ih_1 f$, then there exists a smooth function $\tilde{g}$ such that $X_1(\tilde{g}) = ih_1\tilde{g}$ and $X_2(\tilde{g}) - ih_2\tilde{g} = f$.*

This can be proved keeping track of the proofs of lemmata 6.1 and 6.2 since the condition on f is with respect to one set of variables, x_1 and y_1, whilst the differential equation $X_2(\tilde{g}) - ih_2\tilde{g} = f$ deals only with x_2 and y_2 and treats x_1 and y_1 merely as parameters.

Proposition 7.2. *The first cohomology group $H^1(S_P^\bullet(L))$ vanishes for hyperbolic-hyperbolic singularities.*

Proof. A line bundle valued polarised 1-form, $\alpha \otimes s$, is closed if and only if

$$X_1(\alpha(X_2)) - ih_1\alpha(X_2) = X_2(\alpha(X_1)) - ih_2\alpha(X_1) \ , \tag{44}$$

and it is exact if and only if there exists a smooth function g such that

$$X_j(g) - ih_j g = \alpha(X_j) \ , \tag{45}$$

for $j = 1, 2$.

---

proposition 6.1. Since the aim of this paper is to provide a Poincaré Lemma, this simple computation is left aside.

Let us solve just the first equation,

$$X_1(g) - ih_1 g = \alpha(X_1) \ , \tag{46}$$

using the parametric versions of lemmata 6.1 and 6.2. The closedness of $\alpha \otimes s$ would then imply that

$$X_2 \circ X_1(g) - iX_2(h_1 g) = X_2(\alpha(X_1)) = X_1(\alpha(X_2)) - ih_1\alpha(X_2) + ih_2\alpha(X_1) \ , \tag{47}$$

and because $[X_1, X_2] = X_1(h_2) - X_2(h_1) = 0$ we can write,

$$X_1(X_2(g) - ih_2 g - \alpha(X_2)) = ih_1(X_2(g) - ih_2 g - \alpha(X_2)) \ . \tag{48}$$

Now we can apply lemma 7.1 to prove that there exists a function $\tilde{g}$ such that

$$X_1(\tilde{g}) = ih_1\tilde{g} \tag{49}$$

and

$$X_2(\tilde{g}) - ih_2\tilde{g} = X_2(g) - ih_2 g - \alpha(X_2) \ , \tag{50}$$

then

$$X_j(g - \tilde{g}) - ih_j(g - \tilde{g}) = \alpha(X_j) \ , \tag{51}$$

for $j = 1, 2$.

This proves that the system of equations above has a solution and therefore finishes the proof of this proposition.

□

Now propositions 7.2 and 7.1 entail the following,

Theorem 7.3. *There exists a Poincaré lemma for the Kostant complex in dimension* 4 *for a polarisation given by integrable system in a neighbourhood of an hyperbolic-hyperbolic singularity.*

Together with the results of [22], this theorem asserts that the Kostant complex computes geometric quantisation when the polarisation is given by nondegenerate integrable systems in dimension 4.

7.3 The general case

We can apply the same strategy as in the section above when there are other singularities (the multi-hyperbolic case being the most complicated due to the existence of Taylor flat terms to deal with).

For purely nonhyperbolic singularities, making use of the existence of a torus action, we could apply propositions 4.4 and 6.4 of [22] to deduce

Poincaré lemma from dimension 2 (or 4 if there are focus-focus singularities) to other dimensions. In those cases, we could even give a closed formula combining the explicit homotopy operators in [15] for different circle actions which pairwise commute.[6]

For general singularities we could provide a proof by induction to conclude the following,

Theorem 7.4. *The cohomology groups of the Kostant complex associated to a polarisation defined by an integrable system in a neighbourhood of a singular nondegenerate point vanish in all degrees greater than* 0*.*

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